Exercise 5C: Solutions

1 a Five digits can be arranged in $5! = 120$ ways.

b As the number is odd, there are three possibilities for the last digit (either $1, 3$ or 5). Once this number is selected there are 4 options for the first digit, 3 for the second and so on. This gives a total of $(4 \times 3 \times 2 \times 1) \times 3 = 72$ numbers.

c The first number is 5. There are 4 options for the second digit, 3 for the third and so on. This gives a total of $1 \times 4 \times 3 \times 2 \times 1 = 24$ numbers.

d Without restriction, there are $5! = 120$ numbers. Of these, 24 start with a 5. Therefore, $120 - 24 = 96$ do not begin with a 5.

2 a Five children can be arranged in $5! = 120$ ways.

b Label the three girls with the numbers $1, 2, 3$ and the boys with the numbers 4 and 5. If we group the four boys together we must arrange four objects: $1, 2, 3, \{4, 5\}$. These can be arranged in $4!$ ways. Lastly, the two boys can be arranged in in $2!$ ways. We then use the Multiplication Principle so that there are $4! \times 2! = 48$ arrangements.

c There are 120 arrangements without restriction. Of these, 48 have the boys sitting together. Therefore, $120 - 48 = 72$ will have the boys sitting apart.

d If the boys and girls alternate then the first position must be filled by a girl. This can be filled in 3 ways. The next position must be a boy, and this can be filled in 2 ways. Continuing down the row gives a total of $3 \times 2 \times 2 \times 1 \times 1 = 12$ arrangements.

3 a There are three vowels. Therefore, there are 3 choices for the first letter. Having chosen this letter, there are 5 choices for the second, 4 for the third and so on. This gives $3 \times 5 \times 4 \times 3 \times 2 \times 1 = 360$ permutations.

b There are three vowels. Therefore, there are 3 choices for the first letter. Having chosen this letter, there are 2 choices for the final letter. This leaves 4 choices for the second, 3 for the third and so on. This gives $3 \times 4 \times 3 \times 2 \times 1 \times 2 = 144$ permutations.

c We group the vowels so that we must now arrange four objects: Q,Z,Y,{U,E,A}. This can be done in $4!$ ways. The 3 vowels can be arranged in $3!$ ways. The Multiplication Principle then gives a total of $4! \times 3! = 144$ permutations.

d We group the vowels and consonants together so that we have to arrange two objects: {Q,Z,Y},{U,E,A}. This can be done $2!$ ways. The 3 vowels can be arranged in $3!$ ways and the 3 consonants in $3!$ ways. The Multiplication Principle then gives a total of $2! \times 3! \times 3! = 72$ permutations.

4 a The boys and girls must sit in alternate positions. The 4 boys can be arranged in $4!$ ways and the 4 girls in $4!$ ways. Since the first position is either a boy or a girl we must also multiply by 2. This gives a total of $2 \times 4! \times 4! = 1152$ arrangements.

b We group the boys and girls so that we now have to arrange two groups: {boys}, and {girls}. This can be done in $2!$ ways. The 4 boys can be arranged in $4!$ ways and the 4 girls in $4!$ ways. The Multiplication Principle then gives a total of $2! \times 4! \times 4! = 1152$.

5 a The first digits cannot be 0 so there are only 5 choices for the first digit. The remaining 5 digits can be arranged in $5!$ ways. This gives a total of $5 \times 5! = 600$ arrangements.

b There are two choices for the last digit (0 or 5).
(Case 1) If the last digit is 0 then there are 5 choices for the first digit, 4 for the second and 3 for the third. This gives $5 \times 4 \times 3 \times 1 = 60$ arrangements.
(Case 2) If the last digit is 5 then the first digit cannot be 0. Therefore, there are only 4 choices for the first digit, 4 for the second and 3 for the third. This gives $4 \times 4 \times 3 \times 1 = 48$ arrangements.
This gives a total of $60 + 48 = 108$ arrangements.

c A number less than 6000 will have either $1, 2, 3$ or 4 digits. Note that the first digit can never be 0.

digits	arrangements
1	6
2	$5 \times 5 = 25$
3	$5 \times 5 \times 4 = 100$
4	$5 \times 5 \times 4 \times 3 = 300$

This gives a total of $6 + 25 + 100 + 300 = 431$ different arrangements.

d There are two cases to consider.
(Case 1) If the last digit is 0 then there are 5 choices for the first digit and 4 for the second. This gives $5 \times 4 \times 1 = 20$ arrangements.
(Case 2) If the last digit is 2 or 4 then there are 2 choices for the last digit. The first digit cannot be 0. Therefore, there are only 4 choices for the first digit and 4 for the second. This gives $4 \times 4 \times 2 = 32$ arrangements.
We obtain a total of $20 + 32 = 52$ arrangements.

6 a There are a total of six people to be arranged. This can be done in $6! = 720$ ways.

b There are 2 ways of filling the first position, and there is 1 way of filling the last position. There are 4 choices for the second seat, 3 for the third and so on. This gives a total of $2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$ arrangements.

c Label the two parents with the letters A and B and the children with the letters C, D, E and F. If the children sit together then we have to arrange three objects: $A, B, \{C, D, E, F\}$. This can be done in $3!$ ways. The 4 children can then be arranged in $4!$ ways. This gives a total of $3! \times 4! = 144$ arrangements.

d Label the two parents with the letters A and B and the children with the letters C, D, E and F. If the parents sit together and the children sit together then we first arrange two groups: $\{A, B\}, \{C, D, E, F\}$. This can be done in $2!$ ways. The 2 adults can be arranged in $2!$ ways and the 4 children can then be arranged in $4!$ ways. This gives a total of $2! \times 2! \times 4! = 96$ arrangements.

e Label the two parents with the letters A and B and the children with the letters C, D, E and F, where C is the youngest child. Think of the two parents and the youngest child as one object and there are $4!$ ways of arranging the 4 objects.We also multiply our answer by 2 since the parents can be on either side of the child. This gives a total of $5! \times 2! \times 2 = 480$ arrangements.

7 a As the first digit cannot be 0 there are 9 choices for the first digit. The next digit can now be 0 so there are now 10 choices for the second digit. There are also 10 choices for the third digit. There is only one choice for the fourth and fifth digits. This gives a total of $9 \times 10 \times 10 \times 1 \times 1 = 900$ five-digit palindromic numbers.

b As the first digit cannot be 0 there are 9 choices for the first digit. The next digit can now be 0 so there are now 10 choices for the second digit. There are also 10 choices for the third digit. There is only one choice for the fourth, fifth and sixth digits. This gives a total of $9 \times 10 \times 10 \times 1 \times 1 \times 1 = 900$ six-digit palindromic numbers.

8

The total number of arrangements is $5! = 120$. We now consider those arrangements that begin and end in a vowel. There are 3 choices of vowel for the first position, leaving 2 choices for the last position. The second letter can be chosen 3 ways, the third can be chosen 2 ways and the fourth in 1 way. Therefore, there are $3 \times 3 \times 2 \times 1 \times 2 = 36$ arrangements that begin and end in a vowel.
Finally, the number of arrangements that do not begin and end in a vowel must be $120 - 36 = 84$.

9 **(1-digit numbers)** There are only 2 possibilities.
(2-digit numbers) There are $3 \times 2 = 6$ possibilities.
(3-digit numbers) There are $3 \times 2 \times 2 = 12$ possibilities.
(4-digit numbers) There are $3 \times 2 \times 1 \times 2 = 12$ possibilities.
Adding these gives a total of $2 + 6 + 12 + 12 = 32$ possibilities.

10a The total number of arrangements of six girls is $6! = 720$. We now consider those arrangements where the girls sit together. There are five objects to arrange: $\{A, B\}, C, D, E, F$. This can be done is $5!$ ways. Girls A and B can be arranged in $2!$ ways. This gives a total of $5! \times 2! = 240$ arrangements. Therefore, there are $720 - 240 = 480$ arrangements where the girls do not sit together.

b Assume that girl F goes between girls A and B. We therefore have only four objects to arrange: $\{A, B\}, C, D, E$. Girl F will simply be slotted in-between A and B. Four objects can be arranged in $4!$ ways. Girls A and B can be arranged in $2!$ ways. This gives a total of $4! \times 2! = 48$ arrangements. However, you must multiply this answer by 4 because there are four choices of girl to go between A and B. This gives $48 \times 4 = 192$ arrangements.

11 There are four different patterns possible:

$$\begin{gathered}\text{GBGBGB},\\ \text{BGBGBG},\\ \text{GBBGBG},\\ \text{GBGBBG}.\end{gathered}$$

For each of these patterns, the boys can be arranged in $3!$ ways, and the girls can be arranged in $3!$ ways. This gives a total of $4 \times 3! \times 3! = 144$ arrangements.